Symmetry Groups of Platonic Solids

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Problem

Describe the symmetry groups of the Platonic solids, with proofs, and with an emphasis on the icosahedron.

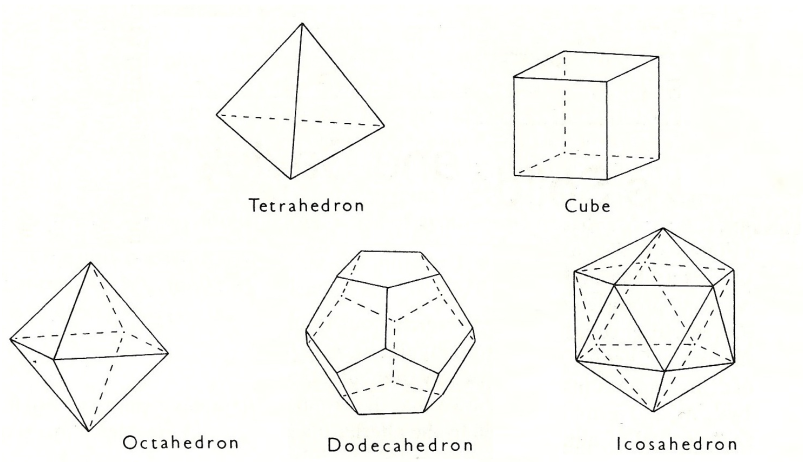

Figure 1: The five Platonic solids.

1 Introduction

The first group of people to carefully study Platonic solids were the ancient Greeks. Although they bear the namesake of Plato, the initial proofs and descriptions of such solids come from Theaetetus, a contemporary. Theaetetus provided the first known proof that there are only five Platonic solids, in addition to a mathematical description of each one. The five solids are the cube, the tetrahedron, the octahedron, the icosahedron, and the dodecahedron. Euclid, in his book *Elements* also offers a more thorough description of the construction and properties of each solid. Nearly 2000 years later, Kepler attempted a beautiful planetary model involving the use of the Platonic solids. Although the model proved to be incorrect, it helped Kepler discover fundamental laws of the universe. Still today, Platonic solids are useful tools in the natural sciences. Primarily, they are important tools for understanding the basic chemistry of molecules. Additionally, dodecahedral behavior has been found in the shape of the earth's crust as well as the structure of protons in nuclei.

The scope of this paper, however, is limited to the mathematical dissection of the symmetry groups of such solids. Initially, necessary definitions will be provided along with important theorems. Next, the brief proof of Theaetetus will be discussed, followed by a discussion of the broad technique used to determine the symmetry groups of Platonic solids. A couple of major ideas which will simplify these proofs are the notions of dual solids and point reflection. To finish, this technique and these ideas will be applied to the specific Platonic solids with a focus on the case of the icosahedron.

2 Defintions and Theorems

Defintion 1 (Platonic solid) *A* **Platonic solid** *is a convex regular polyhedron that satisfies the following three conditions* **(1)** *all its faces are convex regular polygons* **(2)** *none of its faces intersect except at their edges* **(3)** *the same number of faces meet at each vertex.*

Defintion 2 (Dual solid) *The* **dual of a Platonic solid** *is computed by drawing the lines that connect the midpoints of the sides surrounding each polyhedron vertex, and constructing the corresponding tangential polygon.*

Example: Doing this procedure for a cube constructs an octahedron. Therefore the cube and octahedron are **dual solids**. Similarly, the tetrahedron is dual to itself, and the icosahedron is dual to the dodecahedron.

Defintion 3 (Point reflection) *A* **reflection in a point** P *is a transformation of the plane such that the image of the fixed point* P *is* P *and for all other points, the image of* A *is* A' *where* P *is the midpoint of line segment* AA'. *In essence, it is a linear transformation from* $\mathbb{R}^3$ *to* $\mathbb{R}^3$ *which sends each vector* $\mathbf{x}$ *to* $-\mathbf{x}$.

Defintion 4 (Symmetry planes and axes) *Let* X *be an object in Euclidean 3-space. The* **symmetry planes** P *of* X *are essentially mirrors in which the obejct* X *can be reflected while appearing unchanged. The* **symmetry axes** l *of the object* X *are lines about which there exists* $\theta \in (0, 2\pi)$ *such that the object* X *can be brought, by rotating through some angle* θ, *to a new orientation* $X\theta$, *such that* X *appears unchanged.*

Defintion 5 (Symmetry groups) *The* **rotational symmetry group** *of an object X, denoted* $S_r(X)$, *is the group of symmetry of X wherein only rotation is allowed. The* **full symmetry group** *of an object X, denoted S(X), is the group of symmetry of X wherein both rotations and reflections are included.*

Theorem 1 (Theaetetus' Thm) *There are exactly 5 Platonic solids: (1) cube (2) tetrahedron (3) octahedron (4) icosahedron (5) dodecahedron.*

Proof. First note that at least three faces must intersect at any given vertex of a Platonic solid. The incident angles at each vertex must also sum to less than 360° to prevent concavity and flatness. Since Platonic solids are formed by regular polygons, it is necessary to examine what occurs for regular n-gons that are incident at a given vertex.

Now, if $n = 3$, there is an equilateral triangle with regular angle of 60°. In this case, only three, four, or five equilateral triangles may intersect at a given vertex. Anything above six contradicts the definition of a Platonic solid by being either flat or concave. Next, if $n = 4$, there is a square with regular angle of 90°. Exactly three squares can intersect at a given vertex, no more, no less. If more, the solid will violate flatness or concavity. If less, there are not sufficient faces to form a solid. Similarly if $n = 5$, there is a pentagon with regular angle of 108°, and once again exactly three pentagons may intersect at a given vertex, no more, no less. Lastly, if n is greater than 5, all regular angles for these n-gons are greater than or equal to 120°. Therefore the minimum of three necessary incident faces will always produce an angle greater than or equal to 360°, thereby violating the necessary conditions of a Platonic solid.

Thus, there are exactly five Platonic solids because any given Platonic solid is uniquely defined by the number and type of regular polygons which comprise it.

Theorem 2 *Dual solids have the same symmetry group.*

Proof. Since dual polyhedra share the same symmetry planes and symmetry axes, it immediately follows that they share the same symmetry group.

3 Broad Technique

In summary of the main points above, a few things are now clear. Primarily, there are exactly five Platonic solids as seen by **Theorem 1**. This fact, in conjunction with the *Example* and **Theorem 2**, demonstrates that to describe all Platonic solids, it is sufficient simply to describe the symmetry groups of the cube, the tetrahedron, and the icosahedron.

Therefore, to accomplish this task, we must find both the rotational and full symmetry groups for each of the aforementioned Platonic solids: (i) the cube (ii) the tetrahedron and (iii) the icosahedron.

4 Applied Technique

4.1 Cubes and Octahedrons

Proposition 1: *The cube and octahedron both have rotational symmetry groups which are isomorphic to* S_4.

Proof. In order to prove this proposition let us first clarify certain references. As seen in Figure 2 below, let L, M, and N represent various axes of rotation. There are three axes similar to L, which produce nine rotations. There are also six axes similar to M producing six rotations. Finally, there are four main diagonals similar to N, through which the cube may be rotated by 120° and by 240°. This accounts for another eight rotations. Therefore, in addition to the identity element, all of the above rotations sum to twenty-four symmetries.

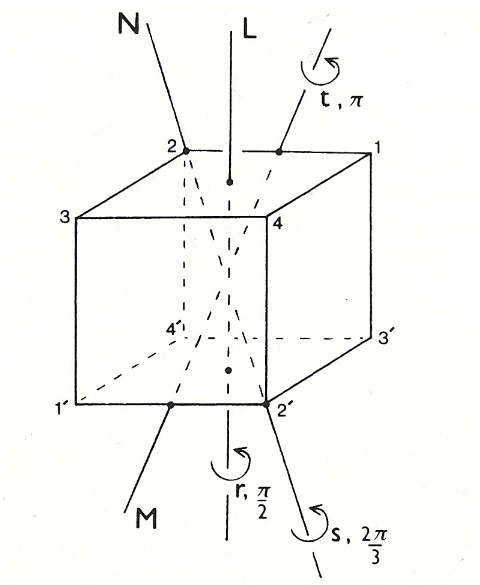

Figure 2: Various axes of the cube.

Observe now that all rotational symmetries of the cube simply permute the four main diagonals of the cube. Now label each corner of the cube as shown in Figure 2, and let N_k denote the diagonal between the point k and k', where $1 \leq k \leq 4$. Each rotational symmetry mentioned before permutes N_1, N_2, N_3, and N_4 which is nothing other than a permutation of the numbers 1, 2, 3, and 4. Referring once again to Figure 1, notice that the rotation r (the rotation anti-clockwise by 90°) sends N_1 to N_2, N_2 to N_3, N_3 to N_4, and N_4 to N_1. This is therefore equivalent to the 4-cycle (1234). The same basic process may be worked out for all other rotations of the cube.

Recall $S_r(C)$ as the rotational symmetry group for the cube C, and denote ϕ: $S_r(C) \rightarrow S_4$ for the function described above. Since we are dealing with rotations, clearly $\phi(\text{xy}) = \phi(\text{x})\phi(\text{y})$ implying that ϕ is a homomorphism. Then, all that remains is checking that ϕ is a bijection. To do this, remember that a surjection between two finite sets which have the same number of elements must be a bijection. We already know $S_r(C)$ has twenty-four elements, and S_4 is known to have twenty-four elements as well. Therefore, we just need to show that ϕ is surjective. Both (1234) and (12) lie in $\phi(S_r(C))$. Additionally, $\phi(S_r(C))$ is a subgroup of S_4 since ϕ sends the multiplication of $S_r(C)$ to that of S_4. Thus, each permutation which can be formed from (1234) and (12) must belong to $\phi(S_r(C))$. The elements (1234) and

(12) are known to generate all of S_4 [**Theorem 6.3**, Armstrong, p. 28]. Therefore we have $\phi(S_r(C)) = S_4$. In conclusion, we have our bijection, and have proven that both the cube and octahedron (thanks to **Theorem 2**) have rotational symmetry groups isomorphic to S_4.

Corollary 1: *The full symmetry group of the cube and octahedron is isomorphic to* $S_4 \times \mathbb{Z}_2$

Proof. This corollary is a direct result of **Proposition 1**, **Theorem 2**, and the definition of **point reflection**. For further explication, refer to [Armstrong, p. 55].

4.2 Tetrahedrons

Proposition 2: *The tetrahedron has rotational symmetry group which is isomorphic to* A_4 *and full symmetry group isomorphic to* S_4.

Proof. Let T be a tetrahedron such that $S_r(T)$ and $S(T)$ are the rotational and full symmetry groups, respectively. Now let us count the number of symmetries for our regular tetrahedron T. Note that there are four positions to which the first vertex may go, by either rotation or reflection. Once we fix the first vertex, then there are three remaining points where the second vertex can go by given rotations. Having fixed the first two vertices, there are still two places for the third vertex to go, due to a reflection. Finally, the last vertex has simply one place left to go. Therefore the total number of symmetries for the tetrahedron is $4 \times 3 \times 2 \times 1$ which is 4!, or twenty-four, total symmetries.

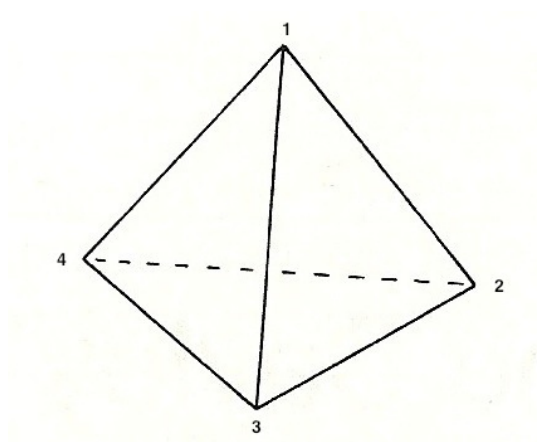

Figure 3: Labeled vertices of the tetrahedron.

Now, let us label each of the vertices with 1, 2, 3, and 4, as seen in Figure 3. The identity then, in cyclic notation is simply (), which is both a reflection and a rotation. The transposition (12) is clearly a reflection. This reflection is performed through the plane containing the center of one edge and the vertex of a face containing the edge. All five other transpositions may be performed in a similar manner. Now, looking at the 3-cycles, we see that (123) is a rotation. This rotation takes the tetrahedron through the symmetry axis that is formed by a vertex and the centroid of a face not containing the vertex, at an angle of 120° in either clockwise or anti-clockwise directions. All seven other 3-cycles may be found by a similar method. Additionally, examining (2,2)-cycles, we find that (12)(34) is a rotation. Here the axis of rotation cuts from a center of an edge through a center of an another edge that is not adjacent to the first edge. The remaining two (2,2)-cycles may be similarly obtained. The only kind of permutation that cannot be generated by a single reflection and a single rotation are the 4-cycles. However, let us examine (1234) as the representative case of all six 4-cycles. Now recognize that (1234) = (12)(13)(14), which involves a reflection of the first kind simply repeated three times.

As before, it remains clear that the function described above is a homomorphism where $\phi(\text{xy}) = \phi(\text{x})\phi(\text{y})$. Therefore, using the same reasoning as in **Proposition 1**, we just need to show that the function ϕ is surjective. However, we did exactly that by matching each element in $S(T)$ to an element in S_4. This proves that $\phi : S(T) \to S_4$ is an isomorphism.

Lastly, notice that solely rotations are mapped to all 3-cycles in S_4 by ϕ. Since 3-cycles generate A_n for all n greater than or equal to 3 [**Theorem 6.5**, Armstrong, p.30], then $\phi(S_r(\text{T})) = A_4$. In conclusion then, we have found that $S(T)$ is isomorphic to S_4 and $S_r(T)$ is isomorphic to A_4.

4.3 Icosahedrons and Dodecahedrons

Proposition 3: *The icosahedron and dodecahedron both have rotational symmetry groups which are isomorphic to* A_5.

Proof. For this last examination of symmetry groups of Platonic solids, we will focus on the case of the icosahedron. To familiarize ourselves with the form, note that an icosahedron has thirty edges, twenty faces, and twelve vertices. We will focus on the symmetry axes which run through the center, joining the midpoints of opposite edges. Since there are thirty edges, there are fifteen such axes. Now, notice that given any one of these axes, there are precisely two other axes that are perpendicular to both the first axis and to each other. Such a grouping of three mutually perpendicular axes will be labeled a **triad**. The fifteen axes are therefore comprised of five sets of triads, which we can label T_1, T_2, T_3, T_4, and T_5. In Figure 4 below, we see the top half of an icosahedron where the endpoints of each triad have been labeled such that points denoted by i correspond to T_i.

Rotating the icosahedron will permute the five triads among themselves, an operation which defines a homomorphism $\phi : S_r(I) \to S_5$, recalling that $S_r(I)$ represents the rotation group of the icosahedron, I. Now, let us compute the order of $S_r(I)$. To begin with, there is the simple identity e. We have also already noted the symmetry axes which run through edges, and there are fifteen of these. The icosahedron can only be rotated through 180° in these edge axes to preserve itself, so the edge axes account for 15×1, or 15 rotations. Next, note that there are six symmetry axes which connect opposite vertices, since there are only twelve vertices. Each axis, however, may be rotated through either 72°, 144°, 216°, or 288° and still preserve the icosahedron. Thus, these axes account for 6×4, or 24 rotations. Lastly, we consider the symmetry axes which run through the twenty faces of the icosahedron. Since these axes cut through opposing faces, there are ten such axes which may be rotated through either 120° or 240°, accounting for the remaining 10×2, or 20 rotations. The sum total of rotations is then $1 + 15 + 24 + 20 = 60$ rotations, giving sixty as the order of $S_r(I)$.

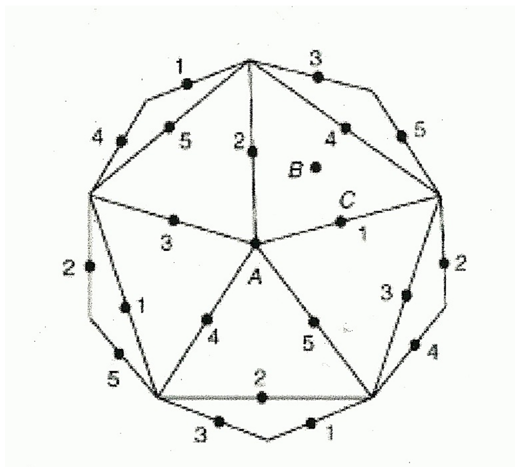

Figure 4: Top half of an icosahedron with labeled endpoints of triads.

As we did in the case of the tetrahedron, we will now relate the rotations of $S_r(I)$ for the icosahedron to the cyclic permutations of S_5. The identity is trivial, relating e to the cyclic notation of (). Now, let us look at the fifteen symmetry axes which run through the midpoints of edges. In Figure 4, these correspond to a rotation about an axis through point C. This particular rotation corresponds to the (2,2)-cycle (23)(45). Taking this as a representative example, we can see that such rotations will be of order 2, and will always be even, since an odd permuation multiplied by an odd permutation is always even [Armstrong, p. 29]. Similarly, recall that an even permuation multiplied by an even permutation is always even. Next, we look at the six symmetry axes which connect opposite vertices. Such rotations may be represented by the rotation through the point A in Figure 4. The rotations about A will correspond to multiples of the 5-cycle (12345). All such multiples will also be even, and extrapolating this representative example to all other vertices demonstrates that all such rotations will correspond to even cyclic permutations. Lastly, the ten symmetry axes which connect faces will be similar to rotations about the axis through point B. These particular rotations about B are multiples of the 3-cycle (142) and are therefore even permuations. All the other rotations through such face axes will be of order three, thereby encompassing all 3-cycles.

As in the previous two cases, it remains clear that the function described above is a well-defined homomorphism where $\phi(\text{xy}) = \phi(\text{x})\phi(\text{y})$ (since we are just dealing with rotations). Thus, using the same reasoning of **Proposition 1** and **Proposition 2**, we simply need to show that ϕ is surjective. However, we did exactly that by matching each element in $S_r(I)$ to an element in A_5, and each group has exactly 60 elements. This proves that $\phi : S_r(I) \to A_5$ is an isomorphism.

Additionally, notice that all rotations are mapped to all 3-cycles in S_5 by ϕ. Since 3-

cycles generate A_n for all n greater than or equal to 3 [**Theorem 6.5**, Armstrong, p.30], then $\phi(S_r(I)) = A_5$. In conclusion, we have found that $S_r(I)$ is isomorphic to A_5, as desired.

Corollary 2: *The full symmetry group of the icosahedron and dodecahedron is isomorphic to* $A_5 \times \mathbb{Z}_2$

Proof. This corollary is a direct result of **Proposition 3**, **Theorem 2**, and the definition of **point reflection**. For further discussion, please refer to [Armstrong, p. 55].

5 Summary

We have now run through the symmetry groups of the Platonic solids. We have proven that the rotational and full symmetric groups of the cube and octahedron, the tetrahedron, and the icosahedron and dodecahedron, are respectively S_4 and $S_4 \times \mathbb{Z}_2$, A_4 and S_4, and A_5 and $A_5 \times \mathbb{Z}_2$.

The savvy use of dual solids nearly halved the amount of work necessary to describe the solids, while a familiarity with the manipulation of cyclic permutations made the rest quite simple. The visual tools included are absolutely necessary for the accessibility of the proofs, and if possible, it is useful to obtain physical representations of the Platonic solids. Being able to look at them from different angles conveys understanding of the symmetry groups in a more concrete fashion.

Lastly, from the work we have put in to understand the Platonic solids, we may further our study and use of the solids. More interesting results, stemming from these symmetry groups, include the study of invariants, Cayley's Theorem and a fuller understanding of Sylow's theorems.

6 References

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[2] H. S. M. Coxeter, *Regular Polytopes*, Dover, New York, 1973.

[3] Pijush K. Ghosh, and Koichiro Deguchi. *Mathematics of Shape Description: A Morphological Approach to Image Processing and Computer Graphics.* John Wiley and Sons, Singapore, 2008.